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Squeezing and non-oriented spin states

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Abstract. We classify the pure states of a spin-s system into oriented and non-oriented states. A pure state is said to be non-oriented if it is not an eigenstate of S^2 and S_z with respect to any axis of quantization. If it is an eigenstate it is oriented. In this paper, we discuss the notion of spin squeezing in these states. Our analysis shows that the oriented states are not squeezed while non-oriented states exhibit squeezing. We also present a new scheme for the construction of spin-s states using $2s$ spinors oriented along different axes. Taking the case of $s = 1$, we show that 'non-oriented' nature and hence squeezing arise from the intrinsic quantum correlations that exist among the spinors in the coupled state.

1. Introduction

The notion of squeezing was initially introduced in the case of a harmonic oscillator [1] and subsequently for the radiation field [2]. Since then it has also been extended to non-canonical systems such as spin. The state of a harmonic oscillator is said to be squeezed if the variance Δx^2 or Δp^2 is less than $\frac{1}{2}$ which is the minimum uncertainty limit. Although squeezing is thus unambiguously defined in the case of bosonic systems [1], its definition in the context of spin needs careful consideration. The components of the spin operator $\vec{S}$ satisfy the commutation relations

$$[S_x, S_y] = \mathrm{i}S_z \qquad x, y, z \quad \text{cyclic} \tag{1}$$

and hence obey the uncertainty relationships

$$\Delta S_x^2 \Delta S_y^2 \geqslant \frac{\langle S_z \rangle^2}{4} \qquad x, y, z \quad \text{cyclic.} \tag{2}$$

A comparison of these uncertainty relations with $\Delta x^2 \Delta p^2 \geqslant \frac{1}{4}$ for a harmonic oscillator would naturally suggest that a spin state could be regarded as squeezed if ΔS_x^2 or ΔS_y^2 is smaller than $\frac{|\langle S_z \rangle|}{2}$, where the expectation value and the variances are calculated in some arbitrary coordinate system. Indeed, this has been used as the squeezing criterion in the literature [2]. This criterion has been critically examined by Kitagawa and Ueda [2] who have remarked that such a definition is coordinate dependent in the sense that a state which is squeezed in a given coordinate frame will not be squeezed in some other coordinate frame. For more details and remarks we refer the reader to [2]. In an attempt to arrive at a proper criterion for squeezing, Kitagawa and Ueda [2] consider the model in which a spin-s state is visualized as being built out of $2s$ elementary spin-$\frac{1}{2}$ states. A coherent spin state (CSS) $|\theta, \phi\rangle$ is then defined as a state in which all the $2s$ elementary spins point in the same direction $\hat{n}(\theta, \phi)$ in real three-dimensional space. Apart from being an eigenstate of $\vec{S} \cdot \hat{n}$ with eigenvalue s,

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this state satisfies the minimum uncertainty relationship: namely (2), with equality sign with uncertainties $\frac{s}{2}$ equally distributed over any two orthogonal spin components normal to the direction $\hat{n}$. As this state is a combination of $2s$ spin-$\frac{1}{2}$ states all pointing in the same direction $\hat{n}$, they conclude that there are no quantum correlations. They then go on to suggest that if quantum correlations are established among the elementary spins, it would be possible to cancel out fluctuations in one direction at the expense of those enhanced in the other direction. Taking this as the physical basis for squeezing, they define that a spin-s state is squeezed if the variance in one spin component orthogonal to the mean spin vector is smaller than the standard quantum limit $\frac{s}{2}$. An alternative criterion [3] for squeezing has also been given by Wineland *et al* . This is based on the requirement of an improvement in sensitivity over what is obtained by using the coherent state in the measurements with spins. Accordingly, a spin-s state is said to be squeezed if the real parameter

$$\xi = \left[\frac{2s(\Delta S_\mu^\perp)^2}{\langle S_\mu\rangle^2}\right]^{\frac{1}{2}} < 1 \tag{3}$$

where $S_\mu^\perp$ is the spin component orthogonal to $\hat{\mu}$. A physical basis for the origin of essentially the above criterion has been obtained by Puri [4] who refers to it as SQII. This criterion, SQII, is claimed to be equivalent to the Kitagawa and Ueda condition referred to as SQI, in the sense that SQII holds only if SQI is satisfied by a state.

In this paper, we study in some detail the structure of spin-s states which are obtained in a novel way from $2s$ elementary spin-$\frac{1}{2}$ states each of which point independently in $2s$ arbitrary directions $\hat{n}_1(\theta_1, \phi_1), \ldots, \hat{n}_{2s}(\theta_{2s}, \phi_{2s})$ and identify those states which exhibit squeezing. We also show that the states so constructed are indeed non-oriented and for $s = 1$, these states possess not only quantum correlations as indicated by Kitagawa and Ueda [2] but also the entangled structure when expressed in terms of the states of the two constituent spinors.

The paper is organized as follows: section 2 deals with the definition of oriented and non-oriented states and their multi-axial nature. The squeezing behaviour of these states is also presented here. In section 3 we present a new scheme of construction of arbitrary spin-s states and use this scheme to construct a non-oriented spin-1 state. We discuss here the correlation aspects associated with the basic spinorial configuration in these states which are shown to be responsible for the manifestation of squeezing.

2. State classification and squeezing

The uncertainty relationship for the components of spin referred to a Cartesian frame xyz with mutually orthogonal directions $\hat{i}$, $\hat{j}$ and $\hat{k}$ is given by

$$\Delta(\vec{S}\cdot\hat{i})^2\Delta(\vec{S}\cdot\hat{j})^2 \geqslant \tfrac{1}{4}\langle\vec{S}\cdot\hat{k}\rangle^2. \tag{4}$$

In order to discuss spin squeezing, we first begin with the squeezing condition itself. Referring to [2–4], we adopt the following definition: *A spin state is squeezed if one of the variances in the spin components normal to the mean spin direction is less than half the modulus of the expectation value of the spin component along the mean spin direction*: i.e., a spin state with $\hat{\mu}$ as the spin direction is said to be squeezed in $S_\mu^\perp$ if

$$\Delta(S_\mu^\perp)^2 < \frac{|\langle S_\mu\rangle|}{2} \tag{5}$$

where $S_\mu^\perp$ is the spin component orthogonal to $\hat{\mu}$.

We first of all start with the familiar angular momentum states $|sm\rangle_{\hat{k}}$: i.e., the eigenstates of S^2 and S_z with respect to the axis of quantization $\hat{k}$. For such states, $\hat{k}$ itself is the mean spin direction and $\langle S_z\rangle = m$. We also have

$$\Delta(\vec{S}\cdot\hat{i})^2 = \tfrac{1}{2}(s(s+1) - m^2) = \Delta(\vec{S}\cdot\hat{j})^2. \tag{6}$$

It is clear that either of the squeezing conditions

$$\Delta(\vec{S}\cdot\hat{i})^2 < \tfrac{1}{2}|\langle\vec{S}\cdot\hat{k}\rangle| \qquad \text{or} \qquad \Delta(\vec{S}\cdot\hat{j})^2 < \tfrac{1}{2}|\langle\vec{S}\cdot\hat{k}\rangle| \tag{7}$$

is not satisfied here for any m. Thus states such as $|sm\rangle_{\hat{k}}$ are not squeezed at all. One can however consider superpositions of the states $|sm\rangle_{\hat{k}}$ of the form

$$|\psi\rangle = \sum_m C_m|sm\rangle_{\hat{k}} \tag{8}$$

and investigate if these exhibit squeezing or not. We consider two mutually exclusive classes of such states which together exhaust all pure states in the $(2s+1)$-dimensional spin space of the system.

2.1. *Oriented spin states*

An oriented spin state by definition is a state $|\psi\rangle$ of the form (8) wherein the coefficients C_m are given by

$$C_m = D^s_{mm'}(\alpha\beta\gamma). \tag{9}$$

Here D^s denote the standard rotation matrices with a fixed index m' and a given set of Euler angles (α, β, γ). In effect, this means that an oriented state $|\psi\rangle$ is an angular momentum state $|sm'\rangle_{\hat{k}'}$ with respect to the quantization axis $\hat{k}'$ in a frame of reference characterized by $\hat{i}'\hat{j}'\hat{k}'$ which is related to the $\hat{i}\hat{j}\hat{k}$ frame via the Euler rotation through α, β, γ. Equation (8) thus takes the form

$$|\psi\rangle = |sm'\rangle_{\hat{k}'} = \sum_m D^s_{mm'}(\alpha\beta\gamma)|sm\rangle_{\hat{k}} \tag{10}$$

for the above class of spin states. To illustrate the significance of the Kitagawa and Ueda condition, we now calculate the relevant variances and the expectation values in the $\hat{i}\hat{j}\hat{k}$ frame which turn out to be

$$\Delta(\vec{S}\cdot\hat{i})^2 = \tfrac{1}{2}(1 - \sin^2\theta\cos^2\phi)[s(s+1) - m'^2] \tag{11}$$

and

$$\langle S_z\rangle = m'\cos\theta \tag{12}$$

where $\alpha = \phi, \beta = \theta, \gamma = 0$ are the Euler angles with θ, ϕ being the polar angles of $\hat{k}'$ with respect to the frame $\hat{i}\hat{j}\hat{k}$.

Having obtained these quantities in an arbitrary frame, we now see that there exists a wide range of values of θ and ϕ for which

$$\Delta(\vec{S}\cdot\hat{i})^2 < \frac{|\langle S_z\rangle|}{2}. \tag{13}$$

However, we cannot call such states squeezed, at this stage, as the mean spin direction is $\hat{k}'$ and not $\hat{k}$. If we now calculate the variances perpendicular to the mean spin direction, they indeed turn out to be exactly equal to those in (6) and thus fail to be squeezed. To appreciate a significant feature of the alternative criterion, that it can be applied in any frame, we find that ξ in the frame $\hat{i}\hat{j}\hat{k}$ itself is given along the x-axis by

$$\xi = \left[\frac{2s(1-\sin^2\theta\cos^2\phi)(s(s+1)-m'^2)}{m'^2\cos^2\theta}\right]^{\frac{1}{2}}. \tag{14}$$

Note that ξ exceeds unity for all m', θ and ϕ except when $\theta = \frac{\pi}{2}$, $\phi = 0, \pi$ in which case it becomes indeterminate. Thus both criteria lead to the same conclusion that no oriented pure state is a squeezed state. In particular, for spin $s = \frac{1}{2}$, it is well known that any arbitrary set of expansion coefficients C_m in (8) may always be identified as (9) with an appropriate choice of α, β, γ, as a consequence of the homomorphism between $SU(2)$ and $O(3)$. Consequently, all pure spin-$\frac{1}{2}$ states are oriented and thus do not exhibit squeezing. This naturally leads us to consider states with $s \geqslant 1$ where there exist states that are intrinsically different from the oriented states. We now turn our attention to such states.

2.2. Non-oriented states

Any normalized spin-s state $|\psi\rangle$ of the form (8) is, in general, specified by $4s$ real independent parameters. The oriented states described above are specified at the most by three parameters which are the three independent Euler angles α, β and γ. Since $4s > 3$, for $s \geqslant 1$, there exist states which are not oriented. In other words, there exist states which cannot be identified as eigenstates of S^2 and S_z with respect to any choice of axis of quantization. We refer to such states as non-oriented. While an oriented state is characterized by a single direction, i.e. the axis of quantization (specified by two real variables θ, ϕ) in the physical space, a non-oriented state could be characterized by more than one direction. In fact, it is interesting to know whether any arbitrary spin state $|\psi\rangle$ specified with respect to some frame in the form (8) is oriented or not. This problem has been studied in quite some detail [5] using the density matrix techniques and the notion of the spherical tensor parameters.

In order to see whether squeezing exists for a non-oriented state we now start with an arbitrary state $|\psi\rangle$ and first determine its mean spin direction $\hat{z}_0$. This can be done, for example, by determining the direction cosines of $\hat{z}_0$ using the expectation values of $\langle\vec{S}\cdot\hat{i}\rangle$, $\langle\vec{S}\cdot\hat{j}\rangle$ and $\langle\vec{S}\cdot\hat{k}\rangle$ in the frame xyz in which $|\psi\rangle$ is initially specified. Once these are obtained, the polar angles of the mean spin direction $\hat{z_0}$ with respect to xyz can be found out. A subsequent rotation is then effected which takes $\hat{k}$ to $\hat{z_0}$. Furthermore, one can also now choose the $\hat{x_0}$ and $\hat{y_0}$ axes conveniently by rotating the resulting frame about $\hat{z_0}$. For the most general spin-1 state that possesses a non-zero mean spin value $\langle\vec{S}\rangle$, such a procedure leads to the form

$$|\psi\rangle = \cos\delta|1,1\rangle_{\hat{z_0}} + \sin\delta|1,-1\rangle_{\hat{z_0}} \qquad 0 < \delta < \pi \tag{15}$$

where $|1, m_0\rangle_{\hat{z_0}}$ are the angular momentum states specified with respect to the $\hat{z_0}$ axis in the $x_0y_0z_0$ frame. This state is obviously non-oriented for all values of δ other than 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, π as the coefficients in (15) do not satisfy (9) for any choice of α, β and γ. For such a state referred to the frame $x_0y_0z_0$, the relevant quantities needed for studying squeezing turn out to be

$$\begin{aligned} \Delta S_{x_0}^2 &= \tfrac{1}{2}(1+\sin 2\delta) \\ \Delta S_{y_0}^2 &= \tfrac{1}{2}(1-\sin 2\delta) \\ \langle S_{z_0}\rangle &= \cos 2\delta \end{aligned} \tag{16}$$

so that the squeezing conditions for S_{x_0} and S_{y_0} are, respectively, given by

$$1 + \sin 2\delta < |\cos 2\delta| \tag{17}$$

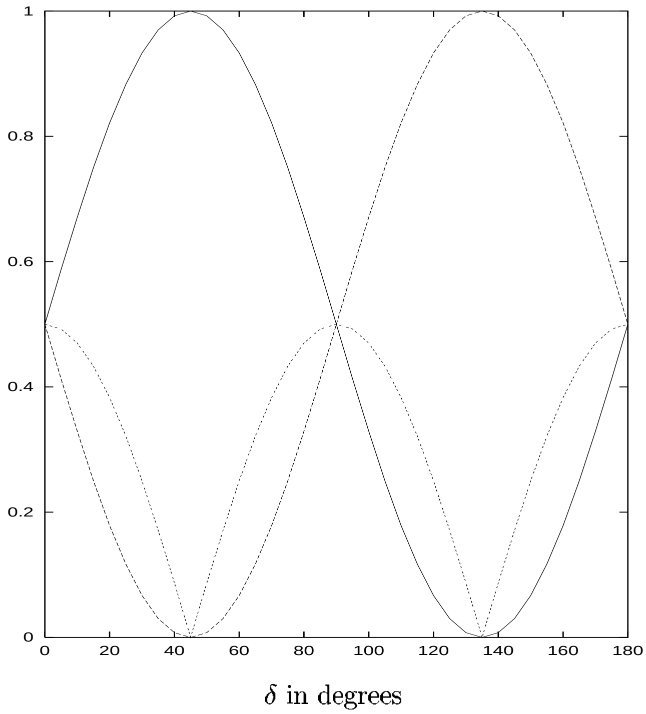

Figure 1. Variations of $\Delta S_{x_0}^2$ (solid curve), $\Delta S_{y_0}^2$(dashed curve) and $|\langle \frac{S_{z_0}}{2} \rangle|$ (dotted curve) with respect to δ are shown. It is clear that except for $\delta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$ (when the state is oriented), the state $|\psi\rangle$ as given by (15) is squeezed either in S_{x_0} or in S_{y_0}.

and

$$1 - \sin 2\delta < |\cos 2\delta|. \tag{18}$$

These conditions are indeed separately valid for the entire range of δ except for $\delta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$ which implies that a non-oriented state $|\psi\rangle$ is indeed a squeezed state. The graph drawn in figure 1, which contains the plots of $\Delta S_{x_0}^2$, $\Delta S_{y_0}^2$ and $\frac{|\langle S_{z_0}\rangle|}{2}$ with respect to δ, indeed show that $|\psi\rangle$ is squeezed in S_{y_0} in the open intervals $0 < \delta < \frac{\pi}{4}, \frac{\pi}{4} < \delta < \frac{\pi}{2}$ and in S_{x_0} in the other intervals: $\frac{\pi}{2} < \delta < \frac{3\pi}{4}, \frac{3\pi}{4} < \delta < \pi$.

Needless to say, the non-oriented state is also squeezed according to the alternative condition, as ξ for the $x(y)$ component given by

$$\xi = \left[\frac{1 \pm \sin 2\delta}{\cos^2 2\delta} \right]^{\frac{1}{2}} \tag{19}$$

is indeed less than unity for the range of values of δ for which squeezing is exhibited according to the former criterion.

3. Quantum correlations

Having thus identified the squeezed states in the spin-1 case, it is of interest to analyse in quantitative terms the suggestion made by Kitagawa and Ueda that squeezing in spin systems arises from the existence of quantum correlations. This can be done by employing the model in which a spin-s state is constructed using $2s$ spin-$\frac{1}{2}$ states as has been suggested by them. Indeed such constructions using spinors were known in mathematics and were defined in their most general form by Cartan [6] much before the spinors were used to describe particles with spin-$\frac{1}{2}$ in physics. Majorana's geometric realization [7] of a spin-s state as a constellation of $2s$ points on a sphere [8] lead to Schwinger's idea [9] of realizing $|sm\rangle$ states in the form,

$$|sm\rangle = \frac{(a_+^\dagger)^{s+m}(a_-^\dagger)^{s-m}}{[(s+m)!(s-m)!]^{\frac{1}{2}}}\,|00\rangle \tag{20}$$

where $a_+^\dagger$, $a_-^\dagger$ are the creation operators for the spin 'up' and spin 'down' states, respectively. It must be noted here that spin 'up' and spin 'down' states as well as $|sm\rangle$ are all referred to the same axis of quantization.

At this point, we would like to generalize this realization by taking $2s$ 'up' spinors $u(\theta_l, \phi_l)$; $l = 1, \ldots, 2s$ where the kth spinor is specified with respect to an axis of quantization $\hat{Q}_k(\theta_k\phi_k)$ in the physical space. Coupling two such spinors leads to

$$|u(\theta_1, \phi_1)u(\theta_2, \phi_2)\rangle = \sum_{m_1, m_2, j} D^{\frac{1}{2}}_{m_1\frac{1}{2}}(\phi_1\theta_1 0)D^{\frac{1}{2}}_{m_2\frac{1}{2}}(\phi_2\theta_2 0)\; C(\tfrac{1}{2}\tfrac{1}{2}j; m_1m_2m)|(\tfrac{1}{2}\tfrac{1}{2})jm\rangle \tag{21}$$

where $j = 0, 1$. Coupling $2s$ spinors this way leads to a spin-s state in the form (8), which implies that the coefficients C_m are given by

$$C_m = N_s d_m \qquad N_s^{-1} = \left\{\sum_{m=-s}^{s} |d_m|^2\right\}^{\frac{1}{2}} \tag{22}$$

where

$$d_m = \sum_{m_1,\ldots,m_{2s-1}} C(\tfrac{1}{2}\tfrac{1}{2}1; m_1m_2\mu_1)C(1\tfrac{1}{2}\tfrac{3}{2}; \mu_1m_3\mu_2)\ldots C(s-\tfrac{1}{2}\tfrac{1}{2}s; \mu_{2s-2}m_{2s}m)\; D^{\frac{1}{2}}_{m_1\frac{1}{2}}(\phi_1\theta_1 0)\ldots D^{\frac{1}{2}}_{m_{2s}\frac{1}{2}}(\phi_{2s}\theta_{2s}0). \tag{23}$$

Thus our construction of a spin-s state $|\psi\rangle$ is done using $2s$ spin-$\frac{1}{2}$ states which are specified with respect to $2s$ different directions: $\hat{Q}_1, \hat{Q}_2, \ldots, \hat{Q}_{2s}$ in general. In particular, if $\hat{Q}_1 = \pm\hat{Q}_2 \cdots = \pm\hat{Q}_{2s}$, then our construction specializes to the realization suggested by Schwinger and employed by Kitagawa and Ueda and Puri and Agarwal. Indeed, in this particular case, the spin state realized is nothing but an oriented state $|sm\rangle$ since the coefficients C_m in equation (22) are identical with (9). The significance of our construction lies in the fact that if $\hat{Q}_l \neq \pm\hat{Q}_m$ for at least two quantization directions, the state realized is a non-oriented state of spin s (a formal proof is given in the appendix). As can be seen from the structure of C_m, the non-oriented states so constructed indeed form a non-denumerable dense subset in the Hilbert space of the spin system.

Considering in particular the simplest case of $s = 1$, we note that such a construction can be carried out using two spinors specified with respect to $\hat{Q}_1(\theta_1\phi_1)$ and $\hat{Q}_2(\theta_2\phi_2)$ so that the spin-1 state

$$|\psi\rangle = N_1 \sum_{m_1,m} D^{\frac{1}{2}}_{m_1\frac{1}{2}}(\phi_1\theta_1 0)D^{\frac{1}{2}}_{m_2\frac{1}{2}}(\phi_2\theta_2 0)C(\tfrac{1}{2}\tfrac{1}{2}1; m_1m_2m)|(\tfrac{1}{2}\tfrac{1}{2})1m\rangle \tag{24}$$

in the lab frame $\hat{i}\hat{j}\hat{k}$ is non-oriented if $\hat{Q}_1 \neq \pm\hat{Q}_2$. The mean spin direction $\hat{z}_0$ for such a state happens to be along the bisector of the two directions $\hat{Q}_1$ and $\hat{Q}_2$. Employing the frame $x_0y_0z_0$ with x_0 lying in the plane of $\hat{Q}_1$ and $\hat{Q}_2$ as shown in figure 2, we see that the polar angles of $\hat{Q}_1$ and $\hat{Q}_2$ with respect to $x_0y_0z_0$ are respectively $(\theta, 0)$ and (θ, π), where 2θ is the angular separation between $\hat{Q}_1$ and $\hat{Q}_2$. The state $|\psi\rangle$ then has the explicit forms

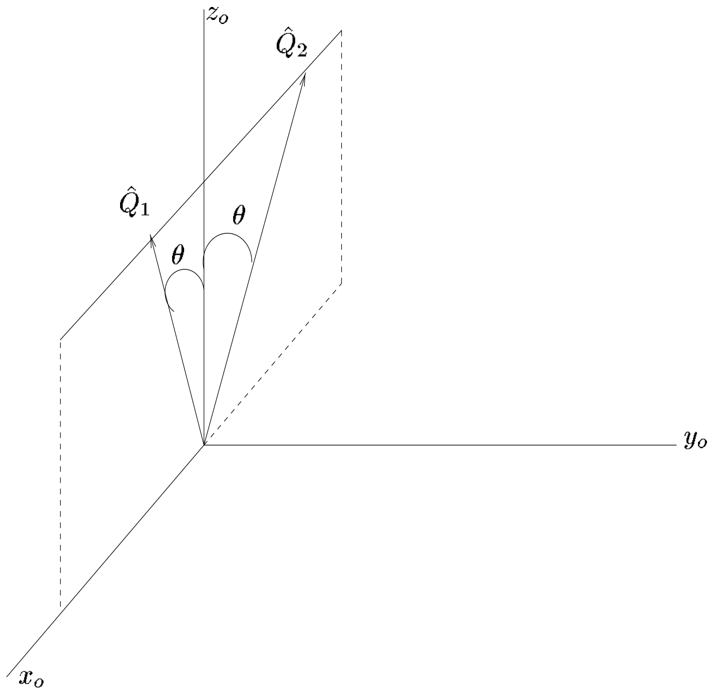

Figure 2. The frame $x_0y_0z_0$ with mean spin direction $\hat{z}_0$ as the bisector of two spinorial quantization axes $\hat{Q}_1$ and $\hat{Q}_2$.

$$|\psi\rangle = -\frac{\mathrm{i}\sqrt{2}}{\sqrt{1+\cos^2\theta}}\left[\cos^2\frac{\theta}{2}\left|\frac{1}{2}\frac{1}{2}\right\rangle_{\hat{z}_0} - \sin^2\frac{\theta}{2}\left|-\frac{1}{2}-\frac{1}{2}\right\rangle_{\hat{z}_0}\right] \tag{25}$$

in terms of the constituent spinor states and

$$|\psi\rangle = -\frac{\mathrm{i}\sqrt{2}}{\sqrt{1+\cos^2\theta}}\left[\cos^2\frac{\theta}{2}|11\rangle_{\hat{z}_0} - \sin^2\frac{\theta}{2}|1-1\rangle_{\hat{z}_0}\right] \tag{26}$$

in terms of the angular momentum states $|1m\rangle_{\hat{z}_0}$ of the spin-1 system. A little insight into the form (25) shows that for $\theta \neq 0, \pi$, $|\psi\rangle$ cannot be written as a simple product of the spin-$\frac{1}{2}$ states implying that it is indeed entangled. The form (26) suggests, on the other hand, that the state is non-oriented for all θ except for $\theta = 0, \frac{\pi}{2}, \pi$. Keeping these observations in mind, we now discuss the squeezing criterion by determining the relevant quantities which turn out to be

$$\Delta S_{x_0}^2 = \frac{1+\cos 2\theta}{2(1+\cos^2\theta)} \tag{27}$$

$$\Delta S_{y_0}^2 = \frac{1}{(1+\cos^2\theta)} \tag{28}$$

and

$$\langle S_{z_0}\rangle = \frac{2\cos\theta}{(1+\cos^2\theta)}. \tag{29}$$

The squeezing condition for S_{x_0} now takes the form

$$\cos^2\theta < |\cos\theta| \tag{30}$$

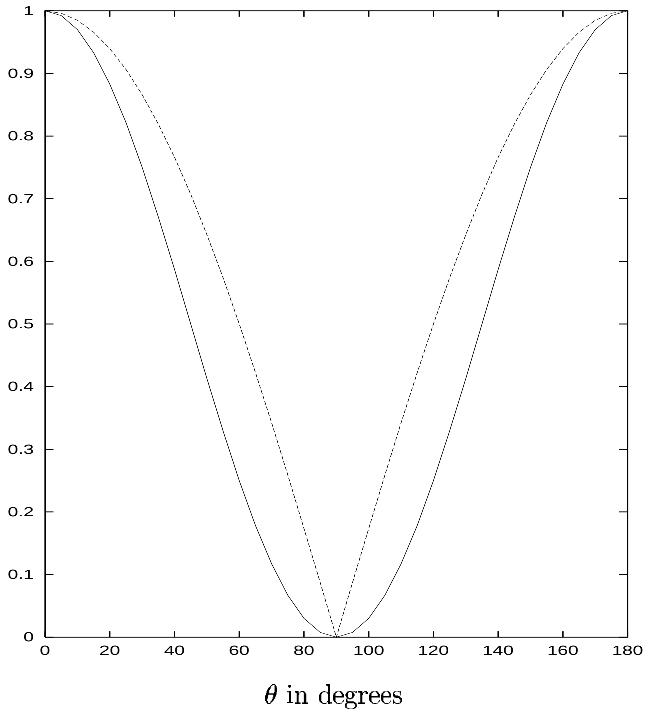

Figure 3. Variations of $\cos^2\theta$ (solid curve) and $|\cos\theta|$ (dotted curve) with respect to θ are shown. The squeezing condition (30) is satisfied for all θ except for $\theta = 0, \frac{\pi}{2}, \pi$.

which is satisfied for all θ except when $\theta = 0, \frac{\pi}{2}, \pi$ as is seen from the graph in figure 3. The absence of squeezing for $\theta = 0, \frac{\pi}{2}, \pi$ is obvious as the two axes then merge together giving an oriented state. Thus in all other cases the state $|\psi\rangle$ is non-oriented by construction and equation (30) implies that every spin-1 non-oriented pure state is squeezed in the spin component S_{x_0} and this squeezing is a function of the angle θ which is half the separation angle between the orientation directions $\hat{Q}_1$ and $\hat{Q}_2$ of the two spin-$\frac{1}{2}$ states.

We now establish explicitly for $s = 1$, the connection between squeezing and the spin–spin correlations that exist between the constituent spinors. Any spin-1 state constructed using the two spinors is said to possess spin correlations if the correlation matrix C^{12} defined through its elements

$$C^{12}_{\mu\nu} = \langle S_{1\mu} S_{2\nu}\rangle - \langle S_{1\mu}\rangle\langle S_{2\nu}\rangle \tag{31}$$

is non-zero. Here $S_{1\mu}$ and $S_{2\nu}$ are the spin components associated with the two spinors and the angular brackets denote the expectation values with respect to the coupled state. For the state $|\psi\rangle$ in (25), the correlation matrix is diagonal in the frame $x_0y_0z_0$ with the 'diagonal' or the 'eigen' elements given by

$$C^{12}_{x_0x_0} = -\frac{\sin^2\theta}{4(1+\cos^2\theta)} = -C^{12}_{y_0y_0} \qquad C^{12}_{z_0z_0} = \left[\frac{\sin^2\theta}{2(1+\cos^2\theta)}\right]^2. \tag{32}$$

A glance at these expressions shows that when $\theta = 0, \frac{\pi}{2}, \pi$, the values of the diagonal entries are either 0 or $\pm\frac{1}{4}$. On the other hand, for all other values of θ, the eigen correlations satisfy

$$0 < |C^{12}_{ii}| < \tfrac{1}{4} \qquad i = x_0, y_0, z_0. \tag{33}$$

In other words, all non-oriented (squeezed) spin-1 states have the eigen correlations restricted to the above range. One can also see that the trace of the correlation matrix is

$$\mathrm{Tr}(C^{12}) = \left[\frac{\sin^2\theta}{2(1+\cos^2\theta)}\right]^2. \tag{34}$$

This being invariant under rotations of the co-ordinate frames, satisfies the condition

$$0 < \mathrm{Tr}(C^{12}) < \tfrac{1}{4} \tag{35}$$

whenever the state is non-oriented (squeezed). Conversely, if a given coupled state has a correlation matrix that satisfies this condition, the state is necessarily non-oriented (squeezed). Given such a correlation matrix, the value of θ can be found out through

$$\cos\theta = \pm\left[\frac{1-2\mathrm{Tr}(C^{12})^{\frac{1}{2}}}{1+2\mathrm{Tr}(C^{12})^{\frac{1}{2}}}\right]^{\frac{1}{2}} \tag{36}$$

which identifies the structure of the state in terms of the two spinors. The four values of θ that satisfy the above equation correspond to two directions: $\pm\hat{Q}_1$ and $\pm\hat{Q}_2$. Thus we conclude that the trace condition (35) on the correlation matrix is the necessary and sufficient condition for a spin-1 state to be squeezed.

4. Summary

In this paper we have classified spin states into oriented and non-oriented states and studied their squeezing properties. Our analysis shows that oriented spin-s states are not squeezed. Considering, in particular, the non-oriented states of a spin-1 system, we have shown that they exhibit squeezing. This has been illustrated in two different ways: first by looking at the non-oriented nature of the spin-1 state itself and secondly, by introducing a new form of coupling in which two spin-$\frac{1}{2}$ states add up to give the required spin-1 non-oriented state. Our construction gives a quantitative description of the existence of quantum correlations as well as an indication as to how they lead to non-oriented nature and hence to squeezing behaviour.

As is clear from our analysis, squeezing is exhibited only by non-oriented states. While in the particular case of spin-1 pure states, the word non-oriented is synonymous with the word 'squeezing', a similar detailed study may be necessary to know whether non-oriented states which are not squeezed exist when $s > 1$. This intimate relationship between squeezing and 'non-oriented' nature indeed suggests that the non-oriented states are experimentally potential candidates for observing squeezing. A recent study by Ramachandran and Deepak [10] reveals that the collision of a spin-$\frac{1}{2}$ beam with a spin-$\frac{1}{2}$ target, both oriented in different directions, leads to a combined spin state which is non-oriented. Another equally interesting aspect is to know how squeezing is exhibited by mixed spin states. A detailed study of this aspect employing the density matrix techniques, is underway.

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Appendix A

Statement. A state $|\psi\rangle$ of spin-s constructed out of $2s$ spinors each with projection $+\frac{1}{2}$ w.r.t $\hat{Q}_i,\ i = 1, \ldots, 2s$ in the form

$$|\psi\rangle = N \sum_{\text{all } m_i} C(\tfrac{1}{2}\tfrac{1}{2}1; m_1 m_2 \mu_1) \ldots C(s - \tfrac{1}{2}\tfrac{1}{2}s; \mu_{2s-2} m_{2s} m) D^{\frac{1}{2}}_{m_1 \frac{1}{2}}(\hat{Q}_1) \ldots D^{\frac{1}{2}}_{m_{2s} \frac{1}{2}}(\hat{Q}_{2s})\ |sm\rangle_{\hat{z}} \tag{A.1}$$

where $\sum_i m_i = m$, N being the normalization constant is oriented if and only if all $\hat{Q}_i$ are collinear.

Proof. If $\hat{Q}_i = \pm\hat{Q}$ for all i, the D's together with the Clebsch–Gordan coefficients can be added to get

$$|\psi\rangle = \sum_m D^s_{mm'}(\hat{Q}) |sm'\rangle_{\hat{z}} \tag{A.2}$$

which is by definition oriented. This proves the sufficiency part. To prove the necessity, we note that if the state $|\psi\rangle$ is oriented as in (A.2), we can choose the $\hat{z}$-axis of the frame along $\hat{Q}$ itself so that $D^s_{mm'}(\hat{Q})$ reduces to $\delta_{mm'}$ indicating that $|\psi\rangle \equiv |sm'\rangle_{\hat{Q}}$ for some fixed m'. Setting $\hat{z} = \hat{Q}$ in (A.1), we see that the above reduction implies that all coefficients of $|sm\rangle_{\hat{Q}}$ except that of $|sm'\rangle$ will be zero. □

The coefficients of each $|sm\rangle$ in (A.1) can be expressed as

$$\sigma_r = \sum_{j=1}^{N} c_{rj} \prod_{i=1}^{n} t_i(j, r) \qquad c_{rj} \neq 0 \qquad \text{for all} \quad r, j$$
$$r = 0, \ldots, n \qquad N = \binom{n}{r} \tag{$E_{n,r}$}$$

where $t_i = p_i = |\cos\frac{\theta_i}{2}|$ for some set of r number of i's and $t_i = q_i = |\sin\frac{\theta_i}{2}|$ for the remaining $(n - r)$ i's. In this form the above analysis implies that if $|\psi\rangle$ is oriented, only one of the σ_r's is non-zero. Note that equation $(E_{n,r})$ simplifies to $\prod_i t_i(1, r) = \frac{\sigma_r}{c_{r1}} = 0$ for $r = 0$ or n (or both). Hence at least one of the $t_i(1, 0 \text{ or } n)$ must be zero; call it t_I. That is, either $p_I = 0, q_I = \sqrt{1 - p_I^2} = 1$ or $q_I = 0$, $p_I = \sqrt{1 - q_I^2} = 1$. Discarding this I and then renumbering the rest of the i's from 1 to $n - 1$, we see that in equation $(E_{n,r})$, the quantity t_I makes its appearance either (i) as $t_I = 0$ in $\binom{n-1}{r-1}$ products $\prod t_i = 0$ and as $\sqrt{1 - t_I^2} = 1$ in the remaining $\binom{n}{r} - \binom{n-1}{r-1} = \binom{n-1}{r}$ products $\prod t_i$; or (ii) vice versa. Incorporating these values we see that equation $(E_{n,r})$, for the surviving $i = 1, 2, \ldots, n - 1$, reappears precisely as equation $(E_{n-1,r})$ or as $(E_{n-1,r-1})$. Repeating the foregoing reasoning we see that at least one other t_i vanishes, and so on. It follows eventually that for every one of the $i = 1, \ldots, 2s$, either $p_i = 0$, $q_i = 1$ or $q_i = 0$, $p_i = 1$. This means that all θ_i are either 0 or π. In other words, all the $\hat{Q}_i$ have to be collinear.

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